Multi-frequency Phase Unwrap from Noisy Data: Adaptive Least Squares Approach

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Abstract. Multiple frequency interferometry is, basically, a phase acquisition strategy aimed at reducing or eliminating the ambiguity of the wrapped phase observations or, equivalently, reducing or eliminating the fringe ambiguity order. In multiple frequency interferometry, the phase measurements are acquired at different frequencies (or wavelengths) and recorded using the corresponding sensors (measurement channels). Assuming that the absolute phase to be reconstructed is piece-wise smooth, we use a nonparametric regression technique for the phase reconstruction. The nonparametric estimates are derived from a local least squares criterion, which, when applied to the multifrequency data, yields denoised (filtered) phase estimates with extended ambiguity (periodized), compared with the phase ambiguities inherent to each measurement frequency. The filtering algorithm is based on local polynomial (LPA) approximation for design of nonlinear filters (estimators) and adaptation of these filters to unknown smoothness of the spatially varying absolute phase [9]. For phase unwrapping, from filtered periodized data, we apply the recently introduced robust (in the sense of discontinuity preserving) PUMA unwrapping algorithm [1]. Simulations give evidence that the proposed algorithm yields state-of-the-art performance for continuous as well as for discontinues phase surfaces, enabling phase unwrapping in extraordinary difficult situations when all other algorithms fail.

Key words: interferometric imaging, phase unwrapping, diversity, local maximum-likelihood, adaptive filtering.
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I. INTRODUCTION

Many remote sensing systems exploit the phase coherence between the transmitted and the scattered waves to infer information about physical and geometrical properties of the illuminated objects such as shape, deformation, movement, and structure of the object's surface. Phase estimation plays, therefore, a central role in these coherent imaging systems. For instance, in synthetic aperture radar interferometry (InSAR), the phase is proportional to the terrain elevation height; in magnetic resonance imaging, the phase is used to measure temperature, to map the main magnetic field inhomogeneity, to identify veins in the tissues, and to segment water from fat. Other examples can be found in adaptive optics, diffraction tomography, nondestructive testing of components, and deformation and vibration measurements (see, *e.g.*, [2], [3]). In all these applications, the observation mechanism is a 2π-periodic function of the true phase, hereafter termed *absolute phase*. The inversion of this function in the interval $[-\pi, \pi)$ yields the so-called *principal* phase values, or *wrapped* phases, or *interferogram*; if the true phase is outside the interval $[-\pi, \pi)$, the associated observed value is wrapped into it, corresponding to the addition/subtraction of an integer number of 2π. It is thus impossible to unambiguously reconstruct the absolute phase, unless additional assumptions are introduced into this inference problem.

Data acquisition with diversity has been exploited to eliminate or reduce the ambiguity of absolute phase reconstruction problem. In this paper, we consider multichannel sensors, each one operating at a different frequency (or wavelengths). Let ψ_s, for $s = 1, \ldots, L$, stand for the wrapped phase acquired by a L-channel sensor. In the absence of noise, the wrapped phase is related with the true absolute phase, φ, as $\mu_s\varphi = \psi_s + 2\pi k_s$, where k_s is an integer, $\psi_s \in [-\pi, \pi)$, and μ_s is a channel depending scale parameter, to which we attach the meaning of frequency. This parameter establishes a link between the absolute phase φ and the wrapped phase ψ_s measured at the s-channel:

$$\psi_s = W(\mu_s \varphi_s) \equiv \mathrm{mod}\{\mu_s \varphi + \pi, 2\pi\} - \pi, \quad s = 1, \ldots, L, \tag{1}$$

where $W(\cdot)$ is the so-called wrapping operator, which decomposes the absolute phase φ into two parts: the fractional part ψ_s and the integer part defined as $2\pi k_s$. The integers k_s are known in interferometry as fringe orders. We assume that the frequencies for the different channels are strictly decreasing, *i.e.*, $\mu_1 > \mu_2 > \cdots > \mu_L$, or, equivalently, the corresponding wavelengths $\lambda_s = 1/\mu_s$ are strictly increasing, $\lambda_1 < \lambda_2 \cdots < \lambda_L$.

Let us mention some of the techniques used for the multifrequency phase unwrap. Multi-frequency interferometry (see, *e.g.*, [11]) provides a solution for fringe order identification using the method of *excess fractions*. This technique computes a set of integers k_s compatible with the simultaneous set of equations $\mu_s \varphi = \psi_s + 2\pi k_s$, for $s = 1, \ldots, L$. It is assumed that the frequencies μ_s do not share common factors, *i.e.*, they are pair-wise relatively prime. The solution is obtained by maximizing the interval of possible absolute phase values. A different approach formulates the phase unwrapping problem in terms of the Chinese remainder theorem, where the absolute phase φ is reconstructed from the remainders ψ_s, given the frequencies μ_s [7], [8]. Statistical modeling for multi-frequency phase unwrapping based on the maximum likelihood approach is proposed in [6]. This work addresses the surface reconstruction from the multifrequency InSAR data. The unknown surface is approximated by local planes. The optimization problem therein formulated is tackled with simulated annealing.

An obvious idea that comes to mind to attenuate the damaging effect of the noise is prefiltering the wrapped observations. We would like, however, to emphasize that prefiltering, although desirable, is a rather delicate task. In fact, if prefiltering is too strong, the essential pattern of the absolute phase coded in the wrapped phase is damaged, and the reconstruction of absolute phase is compromised. On the other hand, if we do not filter, the unwrapping may be impossible because of the noise. A conclusion is, therefore, that filtering is crucial but should be designed very carefully. One of the ways to ensure efficiency is to adapt the strength of the prefiltering according to the phase surface smoothness and the noise level. In this paper, we use the wrapped phase prefiltering technique developed in [9] for a single frequency phase unwrapping. The results presented in this paper is a further elaboration of the maximum likelihood approach proposed and analyzed in [10].

II. PROPOSED APPROACH

We introduce a novel phase unwrapping technique based on local polynomial approximation (LPA) [14] with varying adaptive neighborhood used in reconstruction. We assume that the absolute phase is a piecewise smooth function, which is well approximated by a polynomial in a neighborhood of the estimation point. Besides the wrapped phase, also the size and possibly the shape of this neighborhood are estimated. The adaptive window selection is based on two independent ideas: local approximation for design of nonlinear filters (estimators) and adaptation of these filters to the unknown spatially varying smoothness of the absolute phase. We use LPA for approximation in a sliding varying size window and intersection of confidence intervals (ICI) [14, Ch. 6] for window size adaptation. The proposed technique is a development of the PEARLS algorithm proposed for the single wavelength phase reconstruction from noisy data [9].

We assume that the frequencies μ_s can be represented as ratios

$$\mu_s = p_s / q_s, \tag{2}$$

where p_s, q_s are positive integers and the pairs (p_s, q_t), for $s, t \in \{1, \ldots, L\}$ do not have common multipliers, *i.e.*, p_s and q_t are pair-wise relatively prime.

Let

$$Q = \prod_{s=1}^{L} q_s. \tag{3}$$

Based on the LPA of the phase, the first step of the proposed algorithm computes the least squares estimate of the absolute phase. As a result, we obtain an unambiguous absolute phase estimates in the interval $[-Q \cdot \pi, Q \cdot \pi)$. Equivalently, we get an $2\pi Q$ periodic estimate. The adaptive window size LPA is a key technical element in the noise suppression and reconstruction of this wrapped $2\pi Q$-phase. The complete unwrapping is achieved by applying an unwrapping algorithm. In our implementation, we use the PUMA algorithm [1], which is able to preserve discontinuities by using graph cut based methods to solve the integer optimization problem associated to the phase unwrapping.

The polynomial modeling is a popular idea for both wrapped phase denoising and noisy phase unwrap (e.g. [6], [12], [13], [5]). Compared with these works, the efficiency of the PEARLS algorithm [9] is based on the window size selection adaptiveness introduced by the ICI technique, which locally adapts the amount of smoothing according to the data. In particular, the discontinuities are preserved, what is a sine quo non condition for the success of the posterior unwrapping; in fact, as discussed in [4], it is preferable to unwrap the noisy interferogram than a filtered version in which the discontinuities or the areas of high phase rate have been washed out. In this paper, the PEARLS [9] adaptive filtering is generalized for the multifrequency data. Experiments based on simulations give evidence that the developed unwrapping is very efficient for the continuous as well as discontinuous absolute phase with a range of the phase variation so large that there no alternative algorithms are able to unwrap this data.

III. LOCAL LEAST SQUARED TECHNIQUE

A. Observation Model

Herein, we adopt the complex-valued (cos/sin) observation model

$$u_s = B_s \exp(j\mu_s\varphi) + \sigma n_s,\ \ s = 1, ..., L\ ,\ \ B_s \geq 0, \tag{4}$$

where B_s are amplitudes of the harmonic phase functions, n_s is zero-mean independent complex-valued circular Gaussian random noises of variance equal to 1, *i.e.*, $E\{\mathrm{Re}\, n_s\} = 0$, $E\{\mathrm{Im}\, n_s\} = 0$, $E\{\mathrm{Re}\, n_s \cdot \mathrm{Im}\, n_s\} = 0$, $E\{(\mathrm{Re}\, n_s)^2\} = 1/2$, $E\{(\mathrm{Im}\, n_s)^2\} = 1/2$, and σ is the noise standard deviation. We assume that the amplitudes B_s are non-negative in order to avoid ambiguities in the phase $\mu_s\varphi$, as the change of the amplitude sign is equivalent to a phase change of $\pm\pi$ in $\mu_s\varphi$. We note that the assumption of equal noise variance for all channel is not limitative as different noise variances can be accounted for by rescaling u_s and B_s in (4) by the corresponding noise standard deviation.

Model (4) accurately describes the acquisition mechanism of many interferometric applications, such as InSAR and magnetic resonance imaging. Furthermore, it retains the principal characteristics of most interferometric applications: it is a 2π-periodic function of $\mu_s\varphi$ and, thus, we have only access to the wrapped phase.

B. Phase Estimation

Since we are interested in two-dimensional problems, we assume that the observations are given on a regular $2D$ grid, $X \subset \mathbb{Z}^2$. The unwrapping problem is to reconstruct the absolute phase $\varphi(x, y)$ from the observed wrapped noisy $\psi_s(x, y)$, for $x, y \in X$.

Let us define the parameterized family of first order polynomials

$$\tilde{\varphi}(u, v|\mathbf{c}) = p^T(u, v)\mathbf{c}, \tag{5}$$

where $p = [p_1, p_2, p_3]^T = [1, u, v]^T$ and $\mathbf{c} = [c_1, c_2, c_3]^T$ is a vector of parameters. Assume that in some neighborhood of the point (x, y), the phase φ is well approximated by an element of the family (5); *i.e.,* for (x_l, y_l) in a neighborhood of the origin, there exists a vector $\mathbf{c}$ such that

$$\varphi(x + x_l, y + x_l) \simeq \tilde{\varphi}(x_l, y_l|\mathbf{c}). \tag{6}$$

In [10], we have derived approximate maximum likelihood estimates of $\mathbf{c}$ and $\mathbf{B} \equiv \{B_1, \ldots, B_L\}$ (see (4)). In this paper, we take a slightly different path: we base our inferences in the transformed observation

$$z_s \equiv \exp(j\phi_s) \equiv u_s/|u_s|,\ \ s = 1, ..., L,$$

which can be written in terms of the original phase φ as

$$z_s = \exp(j\mu_s\varphi) + n_s',\ \ s = 1, ..., L. \tag{7}$$

The dependence of model (7) on $\mathbf{B}$ has been moved to the noise terms n_s', which are no more Gaussian distributed.

Motivated by the modified observation model (7), we herein adopt a local least squares criterion to fit the model $\exp(j\mu_s\tilde{\varphi}(x_l, y_l|\mathbf{c}))$ to the observations z_s, where $\tilde{\varphi}$ is the LPA (5) of the original phase φ. The local least squares criterion is given by

$$\begin{aligned} LS_h(\mathbf{c}) &= \frac{1}{2}\sum_s\sum_l \rho_s w_{h,l,s}|z_s(x+x_l, y+y_l) - \exp(j\mu_s\tilde{\varphi}(x_l, y_l|\mathbf{c}))|^2 \\ &\propto -\sum_s\sum_l \rho_s w_{h,l,s}\cos(\phi_s(x+x_l, y+y_l) - \mu_s\tilde{\varphi}(x_l, y_l|\mathbf{c})), \end{aligned} \tag{8}$$

where $w_{h,l,s}$ are window weights parameterized by h, for $h \in H \equiv \{h_1 < h_2 < \cdots < h_J\}$, l is a window relative index, s refers to the wavelength λ_s, and $\rho_s > 0$ are weights of the residuals for the different frequency observations.

A straightforward manipulation of sum in (8) with respect to l leads us to the expression

$$\tilde{L}S_h(\mathbf{c}) = \sum_s \rho_s|F_{w,h,s}(\mu_s c_2, \mu_s c_3)|\cos[\mu_s c_1 - \arg(F_{w,h,s}(\mu_s c_2, \mu_s c_3))] + c^{te}, \tag{9}$$

where c^{te} is an irrelevant constant and $F_{w,h,s}(\mu_s c_2, \mu_s c_3)$ is the windowed Fourier transform of z_s,

$$F_{w,h,s}(c_2, c_3) = \sum_l w_{h,l,s}\, z_s(x+x_l, y+y_l)\exp(-j(c_2 x_l + c_3 y_l)), \tag{10}$$

calculated at the frequencies $(\mu_s c_2,\ \mu_s c_3)$.

C. Phase Estimation for Fixed Windows

Let us study the inference of vector $\mathbf{c}$ for a given window, that is, we are considering a fixed $h \in H$. The phase estimate is defined by optimization over the three variables c_1, c_2, c_3

$$\hat{\mathbf{c}} = \arg\max_{\mathbf{c}} \tilde{L}S_h(\mathbf{c}). \tag{11}$$

Let the condition (2) be fulfilled and $Q \equiv \Pi q_s$. Then, provided fixed c_2 and c_3, the criterion (11) yields a periodic function of c_1 with the period $2\pi Q$. The estimate $\hat{\mathbf{c}}$ obtained according to (11) solves simultaneously two different problems. First, the phase estimate $\hat{c}_1$ is denoised, as it is obtained by a local fitting of a low polynomial model; second, we obtain a partially unwrapping because, in all useful practical applications, we have $Q > 1$.

Because the optimization (11) is computationally quite demanding, we resort to an approximate solution which exploits the equivalence between unconstrained minimization in (11) and the constrained form

$$\min_{c_1}\min_{c_{2,s},c_{3,s}}\sum_s \rho_s|F_{w,h,s}(\mu_s c_{2,s}, \mu_s c_{3,s})|\cos[\mu_s c_1 - \arg(F_{w,h,s}(\mu_s c_{2,s}, \mu_s c_{3,s}))] + c^{te} \tag{12}$$

$$\text{subject to: } c_{2,s} = c_2,\ \ c_{2,s} = c_3 \ \text{ for } \ s = 1, \ldots, L. \tag{13}$$

Our first approximation consists in dropping the constraints (13), which make sense if the unconstrained optimization (9) yields a result close to the constrained optimization (12)-(13). The second approximation consists in assuming that inner maximization in (12), with respect to $c_{2,s}, c_{3,s}$, for $s = 1, \ldots, L$, depends little on the cosine factors. Both approximations make sense if the signal-to-noise is not very low and the true phase is locally well approximated by a plane.

Under these assumptions and approximations, we obtain

$$\hat{c}_1 = \arg\max_{c_1} \tilde{L}S_h(c_1) \tag{14}$$

$$(\hat{c}_{2,s}, \hat{c}_{3,s}) = \arg\max_{c_2,c_3} |F_{w,h,s}(\mu_s c_2, \mu_s c_3)|, \tag{15}$$

where

$$\tilde{L}S_h(c_1) \equiv \sum_s \rho_s|F_{w,h,s}(\mu_s\hat{c}_{2,s}, \mu_s\hat{c}_{3,s})|\cos(\mu_s c_1 - \hat{\psi}_s). \tag{16}$$

with

$$\hat{\psi}_s \equiv \arg[F_{w,h,s}(\mu_s \hat{c}_{2,s}, \mu_s \hat{c}_{3,s})].$$

We note that the absolute phase estimate $\hat{\varphi} = \hat{c}_1$ is calculated by the single-variable optimization (14), whose computational complexity is dominated by the computation of the discrete Fourier transforms $F_{w,h,s}$, for $i = 1, \ldots, L$. By using the fast Fourier transform (FFT), these computations are carried out very efficiently.

For a fixed $h \in H$, the estimates $\hat{\varphi}_s$, for $s = 1, \ldots, L$, are the LPAs of the corresponding wrapped phases φ_s. Note that the objective function $\tilde{L}S_h(c_1)$ is periodical with respect to c_1 with the period $2\pi Q$. Thus, the optimization can be performed only on the finite interval $[-\pi Q, \pi Q)$:

$$\hat{c}_1 = \arg \max_{c_1 \in [-\pi Q, \pi Q)} \tilde{L}S_h(c_1). \tag{17}$$

D. Window Selection

The procedure described in the previous subsection computes, for each pixel, the estimates $\hat{\varphi}_h$ for $h \in H$, where h indexes a set of windows of increasing size. To select, for each pixel, the window size, and thus the corresponding estimate $\hat{\varphi}_h$, we use intersection of confidence intervals (ICI) [14, Ch. 6]. The details of applying ICI to the present phase estimation are presented in [9]. We just stress that the adaptiveness introduced by ICI trades bias with variance in such a way that the window size stretches in areas where the underlying true phase is smooth and shrinks otherwise, namely in the presence of discontinuities.

E. Phase Unwrapping

If the range interval of the absolute phase φ is not larger than $2\pi Q$, the estimate (17) gives a solution of the multifrequency phase unwrap problem, otherwise *i.e.*, the range of the absolute phase φ is larger then $2\pi Q$, then $\hat{c}_1$ gives the phase wrapped to this interval. In this case, a complete unwrapping is obtained by applying one of the standard unwrapping algorithms, as these partially unwrapped data can be treated as obtained from a single sensor modulo-$2\pi Q$ wrapped phase. The above formulas define what we call **LS-MF-PEARLS** algorithm as short for **L**east-**S**quare-**M**ulti-**F**requency **P**hase **E**stimation using **A**daptive **R**egularization based on **L**ocal **S**moothing. The adaptive window size estimates are provided by the PEARL algorithm [9].

IV. EXPERIMENTAL RESULTS

We consider a two-frequency scenario with the wavelength $\lambda_1 < \lambda_2$ and compare it versus a single frequency reconstructions with the wavelengths λ_1 and λ_2 as well as versus the synthetic wavelength $\Lambda_{1,2} = \lambda_1\lambda_2/(\lambda_2 - \lambda_1)$. The measurement sensitivity is reduced when one considers larger wavelengths. This effect can be modelled by the noise standard deviation proportional to the wavelength. Thus, the noise level in the data corresponding to the wavelength $\Lambda_{1,2}$ is much larger than that for the smaller wavelength λ_1 and λ_2.

The proposed algorithm shows a much better accuracy for the two-frequency data than for the data above mentioned corresponding single frequency scenarios. Another advantage of the multifrequency scenario is its ability to reconstruct the absolute phase for continuous surfaces with huge range and large derivatives. The multifrequency estimation implements an intelligent use of the multichannel data leading to effective phase unwrapping in scenarios in which the unwrapping based on any of the data channels would fail. Moreover, the multifrequency data processing allows to successfully unwrap discontinuous surfaces in situations in which the separate channel processing has no chance for success.

In what follows, we present experiments illustrating the LS-MF-PEARLS performance for continuous and discontinuous phase surfaces. For the phase unwrap of the filtered wrapped phase, we use the PUMA algorithm [1], which is able to work with discontinuities. LPA is exploited with the uniform square windows w_h defined on the integer symmetric grid $\{(x, y) : |x|, |y| \leq h\}$; thus, the number of pixels of w_h is $(2h + 1)$. The ICI parameter was set to $\Gamma = 2.0$ and the window sizes to $H \in \{1, 2, 3, 4\}$ and $H = H - 1$ for ML-MF-PEARLS (see [10] for details) and LS-MF-PEARLS algorithms, respectively. The frequencies $(\hat{c}_{2,s}, \hat{c}_{3,s})$ defined in (12) were computed via FFT zero-padded to the size 64×64.

As a test function, we use $\varphi(x, y) = A_\varphi \times \exp(-x^2/(2\sigma_x^2) - y^2/(2\sigma_y^2))$, a Gaussian shaped surface, with $\sigma_x = 10$, $\sigma_y = 15$, and $A_\varphi = 40 \times 2\pi$. The surface is defined on a square grid with integer arguments x, y,

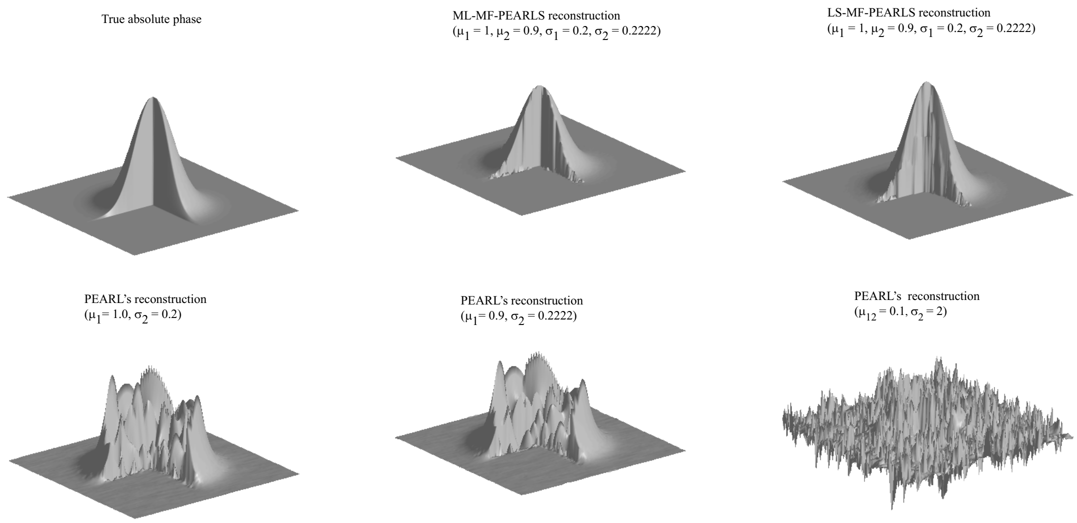

Fig. 1. Discontinuous phase reconstruction. First row: true phase surface; ML-MF-PEARLS reconstruction ($\mu = 9/10$); and LS-MF-PEARLS reconstruction ($\mu = 9/10$). Second row: single frequency PEARLS reconstruction ($\lambda_2 = 1$); single frequency PEARLS reconstruction ($\lambda_2 = 9/10$), and single beat-frequency PEARLS reconstruction ($\lambda_{12} = 1/10$).

$-49 \leq x, y \leq 50$. The maximum value of φ is $40 \times 2\pi$ and the maximum values of the first differences are about 15.2 radians. With such high phase differences, any single channel based unwrapping algorithm fail due to many phase differences larger than π. The noisy observations were generated according to (4), with $B_s = 1$.

We produce two groups of experiments assuming that we have two channels observations with $(\mu_1 = 1, \mu_2 = 4/5)$ and $(\mu_1 = 1, \mu_2 = 9/10)$. Then for the synthetic wavelength $\Lambda_{1,2}$, we introduce the phase scaling factor as $\mu_{1,2} = 1/\Lambda_{1,2} = \lambda_1 - \lambda_2$. For the selected $\mu_1 = 1$ and $\mu_2 = 4/5$, we have $\mu_{1,2} = 1/5$ or $\Lambda_{1,2} = 5$, and for $\mu_1 = 1$ and $\mu_2 = 9/10$, we have $\mu_{1,2} = 1/10$ or $\Lambda_{1,2} = 10$. The period Q corresponding beat wavelength is $\Lambda_{1,2} = 5$, for $\mu_1 = 1$ and $\mu_2 = 4/5$, and $\Lambda_{1,2} = 10$, for $\mu_1 = 1$ and $\mu_2 = 9/10$.

It order to make comparable the accuracy obtained for the signals of different wavelength, we assume that the noise standard deviation is proportional to the wavelength or inverse proportional to the phase scaling factors μ:

$$\sigma_1 = \sigma/\mu_1,\ \sigma_2 = \sigma/\mu_2,\ \sigma_{1,2} = \sigma/\mu_{1,2}, \tag{18}$$

where σ is a varying parameter. Tables I and II show root mean square errors (RMSE) for the following algorithms: PEARLS, introduced in [9], LS-MF-PEARLS (the proposed algorithm), and ML-MF-PEARLS, introduced in [10]). LS-MF-PEARLS and ML-MF-PEARLS shows systematically better accuracy and manage to unwrap the phase when single frequency algorithms fail. Perhaps a bit surprisingly, the performance of the algorithms LS-MF-PEARLS and ML-MF-PEARLS is nearly identical, provided that the sets of the window $H - 1$ and H are used for LS-MF-PEARLS and ML-MF-PEARLS, respectively.

TABLE I

RMSE (IN RAD), $A\varphi = 40 \times 2\pi, \mu_1 = 1, \mu_2 = 4/5$

Algorithm	σ		
	.3	.1	.01
PEARLS, $(\mu_1 = 1)$	fail	fail	fail
PEARLS, $(\mu_2 = 4/5)$	fail	fail	fail
PEARLS, $(\mu_{1,2} = 1/5)$	fail	0.722	0.252
LS-MF-PEARLS	0.588	0.206	0.194
ML-MF-PEARLS	0.587	0.206	0.194

TABLE II
RMSE (IN RAD), $A\varphi = 40 \times 2\pi$, $\mu_1 = 1, \mu_2 = 9/10$

Algorithm	σ		
	.3	.1	.01
PEARLS, $(\mu_1 = 1)$	fail	fail	fail
PEARLS, $(\mu_2 = 9/10)$	fail	fail	fail
PEARLS, $(\mu_{1,2} = 1/10)$	fail	2.268	0.302
LS-MF-PEARLS	0.6759	0.0746	0.0511
ML-MF-PEARLS	0.6718	0.0750	0.0487

We now illustrate the potential in handling discontinuities of bringing together the adaptive denoising and the unwrapping. For the test, we use the Gaussian surface with one quarter set to zero. The corresponding results are shown in Fig. 1. The developed algorithm confirms its clear ability to reconstruct a strongly varying discontinuous absolute phase from noisy multifrequency data.

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