Hybrid Synthesis-Analysis Frame-Based Regularization: A Criterion and an Algorithm

Manya V. Afonso
Instituto de Sistemas e Robótica
Instituto Superior Técnico
1049-001 Lisboa, **Portugal**

José M. Bioucas-Dias and Mário A. T. Figueiredo
Instituto de Telecomunicações
Instituto Superior Técnico
1049-001 Lisboa, **Portugal**

I. Introduction

Consider the problem of estimating a signal/image $\mathbf{x}$ from observations $\mathbf{y}$ that follow the usual linear model $\mathbf{y} = \mathbf{B}\mathbf{x} + \mathbf{n}$, where $\mathbf{B}$ represents a linear observation (e.g., convolution, compressive sensing) and $\mathbf{n}$ is white Gaussian noise. Most frame-based approaches to regularize this inverse problem fall in one of two classes [1], [2]: **(i)** *synthesis* formulations, which are based on representing the unknown image as $\mathbf{x} = \mathbf{W}\boldsymbol{\beta}$, where $\mathbf{W}$ is the synthesis operator of a (tight) frame, and $\boldsymbol{\beta}$ is the vector of representation coefficients, to be estimated by solving the unconstrained convex problem

$$\min_{\boldsymbol{\beta}} \frac{1}{2}\|\mathbf{y} - \mathbf{B}\mathbf{W}\boldsymbol{\beta}\|_2^2 + \tau\,\phi(\boldsymbol{\beta}) \qquad (1)$$

(or a constrained version thereof [5]), where ϕ is a convex sparsity-inducing regularizer (typically, the ℓ_1 norm) and τ its weight; **(ii)** *analysis* formulations, which estimate the image itself (not its representation coefficients) by solving

$$\min_{\mathbf{x}} \frac{1}{2}\|\mathbf{y} - \mathbf{B}\mathbf{x}\|_2^2 + \tau\,\psi(\mathbf{P}\mathbf{x}), \qquad (2)$$

where $\mathbf{P}$ is the analysis operator of a (tight) frame and ψ a convex sparsity-inducing regularizer (usually, also the ℓ_1 norm). If $\mathbf{W}$ is an orthogonal frame, $\mathbf{P} = \mathbf{W}^{-1}$, and $\phi = \psi$, (1) and (2) are equivalent [1]; in general, namely for overcomplete frames, they are not equivalent. Although some debate and research have focused on comparing the two approaches [2], there is no consensus on which of the two is to be preferred. In this paper, we merge the two formulations, by proposing a hybrid synthesis-analysis criterion and an alternating direction algorithm for solving the resulting optimization problem.

II. Proposed Approach

Our hybrid synthesis-analysis criterion is embodied in an unconstrained problem, where the regularizer term is the sum of the synthesis and analysis regularizers from (1) and (2),

$$\min_{\boldsymbol{\beta}} \frac{1}{2}\|\mathbf{y} - \mathbf{B}\mathbf{W}\boldsymbol{\beta}\|_2^2 + \tau_1\,\phi(\boldsymbol{\beta}) + \tau_2\,\psi(\mathbf{P}\mathbf{W}\boldsymbol{\beta}), \qquad (3)$$

where $\mathbf{W}$ and $\mathbf{P}$ are, respectively, the synthesis and analysis operators of two different tight frames (or of the same tight frame; notice that, even in this case, $\mathbf{P}\mathbf{W} \neq \mathbf{I}$). A different hybrid synthesis-analysis (called *balanced*) formulation was recently proposed [3]; however, it requires the analysis and synthesis operators to be of the same frame, thus it is less general.

We attack problem (3) using the variant of the *alternating direction method of multipliers* (ADMM, [4]) that we have proposed in [5] for problems involving the sum of an arbitrary number of convex terms. Each iteration of the algorithm involves applying the Moreau proximity operators of ϕ and ψ (which, if both are ℓ_1 norms, correspond to soft thresholdings), and a least squares minimization, which is efficiently solved, under the following assumptions: $\mathbf{W}$ and $\mathbf{P}$ are, respectively, the synthesis and analysis operators of two Parseval frames ($\mathbf{W}\mathbf{W}^H = \mathbf{I}$ and $\mathbf{P}^H\mathbf{P} = \mathbf{I}$), for which fast transforms exist; $\mathbf{B}$ models a periodic convolution, a subsampling (*i.e.*, we have an inpainting problem), or a partially observed Fourier transform (*i.e.*, one of the classical compressive imaging problems). Finally, we show that sufficient conditions for convergence are satisfied.

III. Experiments and Conclusions

We compare the hybrid formulation with pure synthesis and analysis criteria (solved via the algorithm from [6]), on several benchmark image deconvolution and reconstruction problems (see details of the problems in [6]). For $\mathbf{W}$, we use a 4-level redundant Haar frame; for $\mathbf{P}$, we adopt a 4-level redundant Daubechies-4 frame. Both ϕ and ψ are ℓ_1 norms. To sidestep the issue of adjusting the regularization weights, we simply hand-tune them for maximal ISNR (improvement in SNR); of course, this is inapplicable in practice. Since there is no space in this extended abstract for detailed results, we present a summary of the conclusions drawn from the experiments:

- The analysis and hybrid approaches clearly outperform the synthesis approach in terms of ISNR.
- The synthesis approach reaches its best ISNR faster (by a factor of $5 \sim 10$) than the analysis approach.
- The hybrid approach mildly outperforms the analysis approach in terms of ISNR.
- The hybrid approach reaches its best ISNR faster (by a factor of $2 \sim 3$) than the analysis approach.

Summarizing, the hybrid approach (efficiently handled by the proposed algorithm) yields the best speed/ISNR trade-off: it is preferable to the pure analysis criterion, since it is faster; it is preferable over the synthesis criterion, as it achieves a clearly better ISNR. Of course, these conclusions are based on a limited set of experiments; more work is needed to fully assess the relative merits of these approaches.

References

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