PHASE UNWRAPPING VIA DIVERSITY AND GRAPH CUTS

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ABSTRACT

Many imaging techniques, *e.g.*, interferometric synthetic aperture radar, magnetic resonance imaging, diffraction tomography, yield interferometric phase images. For these applications, the measurements are modulo-$2p$, where p is the *period*, a certain real number, whereas the aimed information is contained in the true phase value. The process of inferring the phase from its *wrapped* modulo-$2p$ values is the so-called *phase unwrapping* (PU) problem. In this paper we present a graph-cuts based PU technique that uses two wrapped images, of the same scene, generated with different periods $p1, p2$. This *diversity* information allows to reduce the ambiguity effect of the wrapping modulo-$2p$ operation, and is extensible to more than two periods. To infer the original data, we assume a first order Markov random field (MRF) prior and a maximum *a posteriori* probability (MAP) optimization viewpoint. The employed objective functionals have nonconvex, sinusoidal, data fidelity terms and a non isotropic total variation (TV) prior. This is an integer, nonconvex optimization problem for which we apply a technique that yields an exact, low order polynomial complexity, global solution. At its core is a non iterative graph cuts based optimization algorithm. As far as we know, all the few existing period diversity capable PU techniques for images, are either far too simplistic or employ simulated annealing, thus exponential complexity in time, optimization algorithms.

1. INTRODUCTION

There are nowadays many applications based on phase images, *e.g.*, interferometric synthetic aperture radar (InSAR) [1], magnetic resonance imaging (MRI) [2], adaptive optics [3], vibration and deformation measurements [4], and diffraction tomography [5]. InSAR is being successfully applied, e.g., to the generation of digital elevation models (DEM), and in the monitoring of land subsidence; among the plethora of MRI applications, we emphasize venography (angiography as well) [6] and tissues elastography [7]; concerning adaptive optics, we point up applications in medicine and industry [8]; interferometry based vibration and deformation measurements is widespread among metrology techniques; diffraction tomography finds application in, *e.g.*, geophysical subsurface prospection and 3D microscopic imaging. In all of these imaging systems, the acquisition sensors read only the cosine and the sine components of the absolute phase; that is, we have access only to the phase modulo-2π, the so-called interferogram. Besides the sinusoidal nonlinearity, the observed data is corrupted by some type of noise. Due to these degradation mechanisms, phase unwrapping is known to be a very difficult problem. In fact, if the magnitude of phase variation between neighboring pixels is larger than 2π, *i.e.*, the so-called Itoh condition [9] is violated, then the inference of the 2π multiples is an ill-posed problem. These violations may be due to undersampling, discontinuities, or noise.

In case we possess two wrapping observations, modulo-$p1$, and modulo-$p2$, of the same signal, by the Chinese remainder theorem [10], under certain circumstances, it is possible to compute the original signal based on those wrapped observations only; the diversity information shortens the ambiguity effect of wrapping observation.

The main contribution of this paper is to present a fast, MAP-MRF based, PU technique that employs the diversity rationale, and to illustrate the boost in the PU performance that the diversity concept brings; the speed enhancement is mainly due to the applied graph-cuts based optimization.

In the next Section we make a quick overview of the diversity concept. Then, in Section 3 we present our technique, namely we will motivate and summarize the chosen graph-cuts algorithm. In Section 4 we show some representative experimental results and, finally, in Section 5 we draw concluding remarks.

2. DIVERSITY

Several signal and image processing techniques employ diversity, which consists of signaling some event or target, *e.g.*, imaging an object, by using diverse relevant parameters. In this paper we consider frequency diversity which is used in various areas, such as, e.g., MRI, echographic Doppler, weather radar, and InSAR.

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Namely, we deal with two (or more) frequencies $F_1 = p/q$, $F_2 = r/s$ where $\{p, q, r, s\} \in \mathbb{N}^1$. For each frequency, we adopt, as in [11], an observation data model to be given by

$$Z_i = e^{jF_i\phi} + n, \tag{1}$$

where ϕ is the true phase, and n is zero-mean, circular, Gaussian noise [11].

For each frequency the log-likelihood function, is $2\pi/F_i$ -periodic. Again referring to [11] we take each log-likelihood to be given by (2)

$$f(\phi|\eta_i) = -\lambda_i \cos(\eta_i - F_i\phi) + c_i, \tag{2}$$

where ϕ stands for the, to be inferred, absolute phase, $\eta_i = angle(Z_i)$ (the observed phases), $\lambda_i \propto |Z_i|$, F_i stand for the employed frequencies, and C_i are irrelevant constants. For our purpose of illustrating diversity PU, we will assume $\lambda_i = 1$, without loss of generality.

For simplicity we use two frequencies only; assuming independence on the random variables that account for the observations (1), it follows that the loglikelihood is given by (3)

$$f(\phi|\eta_1) + f(\phi|\eta_2) = -\cos(\eta_1 - F_1\phi) - \cos(\eta_2 - F_2\phi), \tag{3}$$

we note that $F_1\phi = \eta_1 + 2k\pi$, then with the change of variables $\phi' \equiv F_1\phi$ we get (4)

$$f(\phi'|\eta_1)+f(\phi'|\eta_2) = -\cos(\eta_1-\eta_1+2k\pi)-\cos\left(\eta_2 - \frac{F_2}{F_1}\phi'\right),$$

$$f(\phi'|\eta_1) + f(\phi'|\eta_2) = -1 - \cos\left(\eta_2 - \frac{F_2}{F_1}\phi'\right), \tag{4}$$

in (4) we may discard the -1 term, which leads us to the integer variable k dependent observation model

$$-\cos\left(\eta_2 - \frac{F_2}{F_1}(\eta_1 - 2k\pi)\right), \tag{5}$$

that will be part of the functional to minimize in MAP-MRF framework, as we will refer to in Section 3.

We have already mentioned that the advantage of frequency diversity is to reduce the ambiguity effect of the wrapping modulo-$2p$ operation. Stating it more clearly, it is easy to show that the sum of two cosine functions, having as in (3) different frequencies $F_1 = p/q$ and $F_2 = r/s$, where $\{p, q\}, \{p, r\}, \{q, s\}$, and $\{r, s\}$ are coprime integers[2], results in a third periodic function whose period is $q \times s$; as the initial functions do have periods of respectively q and s, we conclude that the period is extended and, so, the ambiguity reduced. Fig.1 illustrates this effect by plotting the function $f(t) = \cos(t) + \cos(2/5\ t),\ t \in [-8\ \ 8]$ with t in 2π rad units. It can be seen that the period has been extended five times (the initial periods were 2π rad and $5 \times 2\pi$ rad). This "beat production", well known in wave physics, can also be understood by the Chinese remainder theorem [10].

It is a well known behavior, *e.g.*, from wave phenomena, that the greater the beat period extension, the smaller the difference between global and local maxima. Furthermore, it is also well known that beat period extension brings noise amplification. This trade-off should then be taken into account.

[1]Rigorously, F_1 and F_2 can be irrational as long as their quotient is rational. However this does not take any generality in what follows.

[2]Two integer numbers are said to be coprime if their greatest common divisor is the unity.

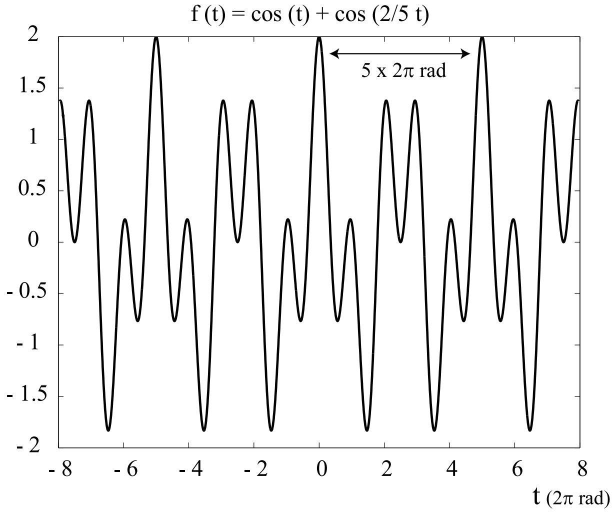

Fig. 1. Ambiguity reduction by summing two periodic functions: the beat effect.

3. PROPOSED ALGORITHM

3.1. Related work

Definitely, frequency diversity based PU techniques are scarce. We are aware only of [12], [13], and [14] published in 1994, 1998 and 2002, respectively. Regarding the first [12] it proposes three very simple (and interesting) algorithms that, nonetheless, are error prone. With respect to the second [13], it is a multidimensional (accounting for multifrequency) version of the minimum L^2 norm type of PU algorithm [15], with relaxation to the continuum that is well-known [15] to give rise to solving a Poisson equation. The weaknesses of this approach are long-familiar, in particular the oversmoothing of high phase rate slopes and discontinuities, which is further amplified by the proposed previous low-pass filtering stage (see [11] for a deeper discussion on this problem). Concerning [14], it consists of an algorithm based on a maximum likelihood estimation technique, whose goal is to approximate the unknown (true) surface by means of local planes. The approach assures the uniqueness of the solution even accounting for high phase rate slopes or discontinuities. However, the global optimization required to compute the maximum likelihood, by suggestion of the authors, is to be achieved by simulated annealing, which is a too much slow optimization technique to tackle this problem, for which, *e.g.*, graph-cuts techniques are much more suited.

3.2. Graph-cuts formulation

Let us consider the undirected graph $(\mathcal{V}, \mathcal{E})$ where the set of nodes $\mathcal{V}$ represents image pixels and the set of edges $\mathcal{E}$ represents pairs of neighboring pixels (horizontal and vertical in our case).

In Section 2, the loglikelihood expression (5) was obtained based on a bayesian model ([11]) and for two sources (two frequencies). Considering that the prior is a MRF defined on $(\mathcal{V}, \mathcal{E})$, then the logarithm of the posterior density is, in our case, given by

$$\begin{aligned} E(k) \;&\equiv\; \sum_{i\in\mathcal{V}} -\cos\left(\eta_{2_i} - \frac{F_2}{F_1}(\eta_{1_i} - 2k_i\pi)\right) \\ &+\; \mu \sum_{(i,j)\in\mathcal{E}} V_{ij}\left(\phi'_i, \phi'_j\right), \end{aligned} \tag{6}$$

where $\phi' = (\phi'_1, \phi'_2, \ldots, \phi'_{|\mathcal{V}|})$, $\phi' = F_1\phi = \eta_1 + 2k\pi$, we take $V_{ij}\left(\phi'_i, \phi'_j\right) = |k_i - k_j|$, the so-called non-isotropic TV, and finally μ is the regularization parameter that sets the relative weight between the data fidelity and the prior terms.

We are aware of only three integer optimization algorithms, that are able to provide a global minimum for a posterior energy like (6), which is composed by a non-convex data fidelity term and a convex prior potential. Those algorithms were introduced in [16], [17], and [18]. Herein, we refer to the last one [18]. As long as the energy is a levelable function, i.e., a function that admits a decomposition as a sum on levels, of functions of its variables level-set indicatrices at current level (see [18]), it is easy to build a graph such that its maxflow is the sought global minimum. For the sake of simplicity we do not describe the above mentioned energy decomposition on level-set dependent functions, as we will not describe the graph construction. We remark only that there exist plenty of low order polynomial complexity maxflow algorithms, and it is simple to proof that non-isotropic TV is a levelable function.

4. EXPERIMENTAL RESULTS

Fig. 2 illustrates the effectiveness of the proposed approach to phase unwrapping (we have employed a regularization parameter $\mu = 0.1$). In Figs. 2 (a) and 2 (b) we show the images of a wrapped gaussian having a $50 \times \pi$ height peak. Each of those wrapped images is generated according to an InSAR observation statistics (see, e.g., [11]) and with gaussian noise (Signal to noise ratio (SNR) of 10 dB); they differ in the relative signal frequency employed in each of them which is $F_1 = 1$ and $F_2 = 4/5$ respectively. An aliasing effect is quite clear in both. In Fig. 2 (c) we show the unwrapping result by limiting the signal excursion to five 2π levels only. It can be seen that with five levels (five is the ambiguity interval extension factor used) there is no more aliasing. In Fig. 2 (d) we show the result of the our proposed diversity graph cuts based algorithm; it can be seen that the unwrapping is perfect even in the presence of noise (SNR $= 10$ dB). Furthermore the result was obtained in a few seconds on a 2.0 GHz dual core Pentium.

5. CONCLUDING REMARKS

The presented results illustrate and indicate that frequency diversity can greatly enhance the phase unwrapping performance. Furthermore, the employed graph-cuts global optimization in the MAP-MRF scenario proves to be very fast. Given the growing importance of phase imaging techniques diversity research should be prosecuted. In future work we intend to explore generalized kinds of diversity like temporal PU [19] and more generally multidimensional PU.

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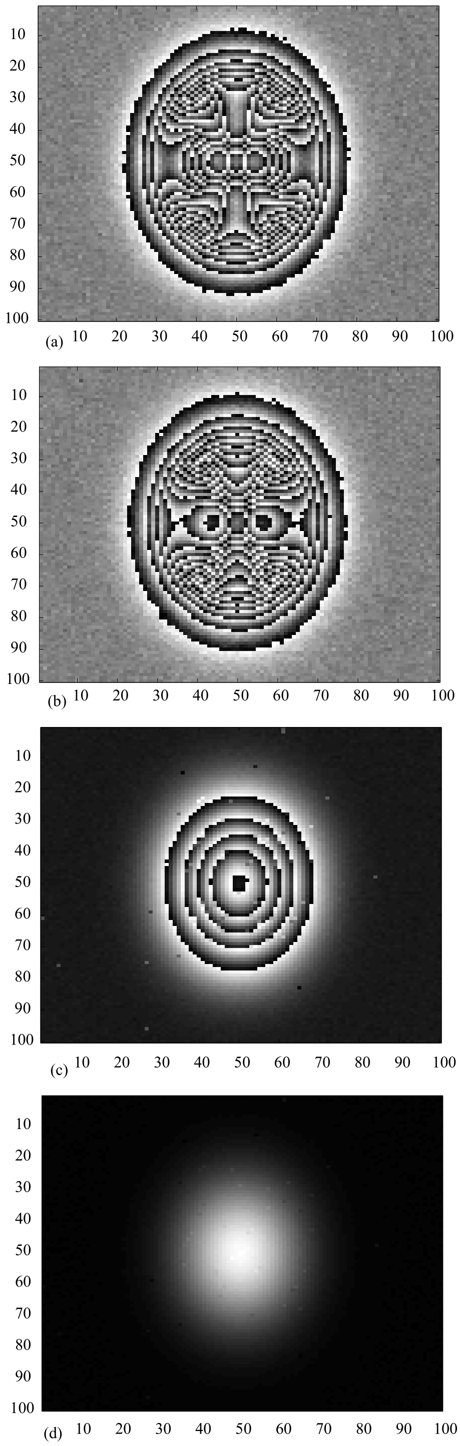

Fig. 2. (a) Wrapped Gaussian elevation (50π rad); the image was obtained with a relative frequency of 1 and SNR$=10$ dB. (b) Wrapped Gaussian elevation (50π rad); the image was obtained with a relative frequency of $4/5$ and SNR$=10$ dB. (c) Images in (a) and (b) unwrapped but limited to the ambiguity interval ($\mu = 0.1$); no aliasing. (d) Images in (a) and (b) unwrapped by our proposed algorithm ($\mu = 0.1$); perfect unwrapping even with noise.

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