A ROBUST SUBSPACE METHOD FOR SEMIBLIND DICTIONARY-AIDED HYPERSPECTRAL UNMIXING

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ABSTRACT

Recent development in semiblind dictionary-aided hyperspectral unmixing (HU) shows that a classical method in sensor array processing, namely, multiple signal classification (MUSIC), provides an effective way for endmember identification. However, MUSIC (and in fact, other dictionary-based sparse regression algorithms) assumes that there are no mismatches between the true endmember signatures and the dictionary spectral signatures, which may be violated in practice owing to reasons such as endmember variability and calibration errors. This paper presents a robust MUSIC method, wherein spectral signature mismatches are incorporated in the original MUSIC formulation to make the resulting algorithm robust. A computationally simple method is derived for the implementation of robust MUSIC. Simulation results show that robust MUSIC provides improved robustness against spectral signature mismatches than the original MUSIC.

***Index Terms*—** hyperspectral unmixing, dictionary, robust method, subspace method

1. INTRODUCTION

A spectral library is a collection of spectral signatures of materials acquired in controlled or ideal environments, e.g., in laboratories. There are several available libraries, provided by government agencies and research institutes. For example, the the U.S. Geological Survey (U.S.G.S.) library [1] contains remotely sensed and extracted spectral signatures of over 1300 materials.

The rich knowledge of materials' spectra in the existing libraries provides new opportunities for hyperspectral umixing (HU). By using an existing library as a dictionary, and by assuming the linear mixture model, we can formulate HU as a problem of selecting spectra from the dictionary to fit the mixture model. Such a semiblind dictionary-aided HU problem is fundamentally identical to the well-known basis selection or sparse regression problem in compressive sensing (CS), and thus many CS tools can be applied for tackling the HU problem. There are several advantages with semiblind dictionary-based HU. First, different from the completely blind HU approaches (e.g., [2, 3]), dictionary-based methods do not require certain restrictive assumptions, such as the pure-pixel assumptions, to perform HU. Second, dictionary-based methods may not require knowledge of the number of endmembers.

Dictionary-based HU algorithms were proposed in [4–7]. Among them, we are interested in the *multiple signal classification* (MUSIC) method [7]—which is an application of a classical sensor array processing approach to the frontier semiblind dictionary-aided HU problem. MUSIC is a closed-form-based algorithm, and is computationally efficient to implement. Under some ideal assumptions, MUSIC can perfectly identify the true endmembers. In practice, MUSIC proves to be very useful as a pruning method for preselecting candidates from the dictionary, so that a smaller size and more representative dictionary can be constructed for another sparse regression algorithm to perform semiblind HU.

While dictionary-based HU is an appealing approach, there are challenges. In practice, the spectral signature samples in the dictionary may not exactly match the groundtruth endmembers' spectral signatures in the scene. There are several reasons for this. First, the materials' spectra may vary from time to time and site to site subject to diverse physical restrictions, e.g. strength of sunlight and temperature [8]. Second, the calibration procedure for spectral signatures may introduce errors. Third, the spatial resolutions of spectra in the dictionary can be different from those of the image, and that can also result in modeling errors. MUSIC is particularly sensitive to spectral signature mismatches.

In this work, we develop a robust MUSIC method for tackling the spectral signature mismatch problem. We first give a robust MUSIC formulation, wherein the goal is to identify spectral signature samples that are close to true endmember signatures, rather than exactly equal. Such an objective is meaningful since such identified spectral signatures, albeit inexact, can still provide useful information, such as the classes of the materials in the image. Also, as mentioned above, such detected spectra can be used as a pruned dictionary for sparse regression. Then we derive a simple-to-compute algorithm based on the proposed robust MUSIC formulation. Simulations are used to show the effectiveness of the proposed algorithm.

2. PROBLEM STATEMENT

To describe semiblind dictionary-aided HU, let

$$\boldsymbol{A}_D = [\, \boldsymbol{a}_1, \ldots, \boldsymbol{a}_K \,]$$

denote a given spectral library, where each $\boldsymbol{a}_k \in \mathbb{R}^M$, $k = 1, \ldots, K$, is a spectral signature of a particular material, M is the number of spectral bands, and K is the dictionary size. The number K is often large (e.g., more than 1300 for the U.S.G.S. library), and thus $\boldsymbol{A}_D$ is over-complete. We are given a measured hyperspectral data set $\boldsymbol{y}[1], \ldots, \boldsymbol{y}[L]$, where $\boldsymbol{y}[\ell] \in \mathbb{R}^M$ denotes the measured spectral vector at pixel ℓ, and L is the number of pixels.

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Let $\boldsymbol{Y} = [\ \boldsymbol{y}[0], \ldots, \boldsymbol{y}[L-1]\ ]$. Under the linear mixing model, the data matrix $\boldsymbol{Y}$ can be modeled as

$$\boldsymbol{Y} = \boldsymbol{A}\boldsymbol{S} + \boldsymbol{V}, \tag{1}$$

where$\boldsymbol{S} = [\ \boldsymbol{s}[0], \ldots, \boldsymbol{s}[L-1]\ ]$; $\boldsymbol{s}[\ell] = [\ s_1[\ell], \ldots, s_N[\ell]\ ]^T$, $s_n[\ell] \geq 0$ for all $n = 1, \ldots, N$, is the abundance vector at pixel ℓ; N is the number of endmembers; $\boldsymbol{V} = [\ \boldsymbol{v}[0], \ldots, \boldsymbol{v}[L-1]\ ]$ is noise; $\boldsymbol{A} \in \mathbb{R}^{M \times N}$ is the endmember matrix. In semiblind dictionary-aided HU, we assume that every column of $\boldsymbol{A}$ is covered by the dictionary; i.e.,

$$\boldsymbol{A} = [\ \boldsymbol{a}_{k_1}, \ldots, \boldsymbol{a}_{k_N}\ ], \tag{2}$$

$$k_n \in \{\ 1, \ldots, K\}, \quad n = 1, \ldots, N. \tag{3}$$

The problem is as follows: Given the dictionary $\boldsymbol{A}_D$, identify the index set $\Lambda = \{k_1, \ldots, k_N\}$ from the measured data $\boldsymbol{Y}$. Note that once Λ is identified, we can use it to construct the endmember matrix and then perform unmixing. In this work, the number of endmembers N is assumed to be known, or has been acquired by other algorithms such as HySiMe [9].

3. REVIEW OF THE MUSIC APPROACH

The index set selection problem mentioned in the last section can be tackled by a subspace method called MUSIC [7, 10]. The method is briefly described as follows. Under the assumption that $|\Lambda| < \mathrm{spark}(\boldsymbol{A}_D) - 1$, where $\mathrm{spark}(\boldsymbol{X}) = r$ means that any r columns of $\boldsymbol{X}$ are linearly independent, we have

$$\boldsymbol{P}_{\boldsymbol{A}}^{\perp} \boldsymbol{a}_k = \boldsymbol{0} \Leftrightarrow k \in \Lambda, \tag{4}$$

where $\boldsymbol{P}_{\boldsymbol{X}}^{\perp}$ denotes the orthogonal complement projector of $\boldsymbol{X}$. In practice, if $\mathrm{rank}(\boldsymbol{A}) = \mathrm{rank}(\boldsymbol{S}) = N$, a basis of $\mathcal{R}(\boldsymbol{A})$, denoted by $\boldsymbol{U}_S \in \mathbb{R}^{M \times N}$, can be estimated by principal component analysis (PCA) or HySiMe. Hence, we can identify the endmember index set by the following procedure:

1. For $k = 1, \ldots, K$, calculate

$$\gamma_{\mathrm{MUSIC}}(k) = \frac{\boldsymbol{a}_k^T \boldsymbol{P}_{\boldsymbol{U}_S}^{\perp} \boldsymbol{a}_k}{\|\boldsymbol{a}_k\|_2^2}. \tag{5}$$

2. Determine $\hat{\Lambda} = \{k_1, \ldots, k_N\}$ such that for $n = 1, \ldots, N$, we have $\gamma_{\mathrm{MUSIC}}(k_n) < \gamma_{\mathrm{MUSIC}}(j)$ for all $j \notin \hat{\Lambda}$.

The above procedure is called MUSIC. Notice that in (5), the denominator $\|\boldsymbol{a}_k\|_2$ is introduced to avoid numerical problems caused by scalings of $\boldsymbol{a}_k$'s.

As can be seen above, MUSIC is easy to implement. Under some ideal assumptions, e.g., no noise, MUSIC is shown to guarantee perfect identification of Λ. In practice, it was found that MUSIC is useful for dictionary pruning; please see [7] for the details.

4. ROBUST MUSIC FOR IMPERFECT DICTIONARIES

The crucial assumption with dictionary-based HU is that for each endmember, the dictionary has a spectral signature sample that exactly matches the true endmember spectral signature. As discussed in Introduction, this may be not the case in reality. Herein we consider a spectral signature mismatch model, and then develop a robust subspace method for tackling the issue. To this end, we modify the signal model in (1) as

$$\boldsymbol{Y} = \boldsymbol{B}\boldsymbol{S} + \boldsymbol{V}, \tag{6}$$

where $\boldsymbol{B} = [\ \boldsymbol{b}_1, \ldots, \boldsymbol{b}_N\ ]$ now denotes the endmember matrix. We assume that there are errors between an endmember signature $\boldsymbol{b}_n$ and its corresponding dictionary sample. To be specific, we model

$$\boldsymbol{a}_{k_n} = \boldsymbol{b}_n + \boldsymbol{e}_n, \tag{7}$$

for some $k_n \in \{1, \ldots, K\}$, where $\boldsymbol{e}_n$ is an error vector. The spectral error vectors are assumed to be bounded:

$$\|\boldsymbol{e}_n\|_2 \leq \epsilon, \quad n = 1, \ldots, N,$$

for some $\epsilon > 0$.

The robust MUSIC method is proposed as follows. We robustify the MUSIC procedure in the last section by replacing $\gamma_{\mathrm{MUSIC}}(k)$ in (5) with

$$\gamma_{\mathrm{RMUSIC}}(k) = \min_{\boldsymbol{\xi} \in \mathbb{R}^M} \frac{(\boldsymbol{a}_k - \boldsymbol{\xi})^T \boldsymbol{P}_{\boldsymbol{U}_S}^{\perp} (\boldsymbol{a}_k - \boldsymbol{\xi})}{\|\boldsymbol{a}_k - \boldsymbol{\xi}\|_2^2} \tag{8a}$$

$$\text{s.t. } \|\boldsymbol{\xi}\|_2 \leq \delta, \tag{8b}$$

for a given parameter $\delta > 0$. The rationale of the robust MUSIC (RMUSIC) problem above is to find a vector $\boldsymbol{\xi}$ to compensate the error term $\boldsymbol{e}_n$, thereby attempting to make the method more resilient to spectral signature mismatches. An illustrative example is shown in Fig. 1 to demonstrate why RMUSIC may be more preferable than MUSIC. One can see in the example that if one applies MUSIC directly, $\boldsymbol{a}_j$ (when scaled to the unit 2-norm circle) results in a smaller projection residual (i.e., $\gamma_{\mathrm{MUSIC}}(j)$) than that of the desired $\boldsymbol{a}_k$. However, by using RMUSIC, this confusion can be fixed.

The key issue with realizing RMUSIC lies in solving problem (8). Problem (8) is a *single-ratio quadratic fractional program*, which is quasi-convex and can be optimally solved, e.g., by the *Dinkelbach algorithm* and its variants [11]. While this means that we can implement RMUSIC by existing optimization algorithms, we have to solve K such quasi-convex problems—which can still be inefficient for large K (often true in practice). To circumvent this difficulty, we consider a compromise by approximating $\gamma_{\mathrm{RMUSIC}}(k)$ in (8) via a simple-to-compute lower bound. To derive the lower bound, we first re-express $\gamma_{\mathrm{RMUSIC}}(k)$ as

$$\begin{aligned}\gamma_{\mathrm{RMUSIC}}(k) &= \min_{\|\boldsymbol{\xi}\|_2 \leq \delta} \frac{\left\|\boldsymbol{P}_{\boldsymbol{U}_S}^{\perp}(\boldsymbol{a}_k - \boldsymbol{\xi})\right\|_2^2}{\|\boldsymbol{P}_{\boldsymbol{U}_S}^{\perp}(\boldsymbol{a}_k - \boldsymbol{\xi})\|_2^2 + \|\boldsymbol{P}_{\boldsymbol{U}_S}(\boldsymbol{a}_k - \boldsymbol{\xi})\|_2^2} \\ &= \min_{\|\boldsymbol{\xi}\|_2 \leq \delta} \frac{\eta_k^2(\boldsymbol{\xi})}{\eta_k^2(\boldsymbol{\xi}) + 1},\end{aligned} \tag{9}$$

where $\boldsymbol{P}_{\boldsymbol{U}_S}$ denotes the projector on to $\mathcal{R}(\boldsymbol{U}_S)$, and

$$\eta_k(\boldsymbol{\xi}) = \frac{\left\|\boldsymbol{P}_{\boldsymbol{U}_S}^{\perp}(\boldsymbol{a}_k - \boldsymbol{\xi})\right\|_2}{\left\|\boldsymbol{P}_{\boldsymbol{U}_S}(\boldsymbol{a}_k - \boldsymbol{\xi})\right\|_2}. \tag{10}$$

Since the objective function of (9) is a monotone increasing function of $\eta^2(\boldsymbol{\xi}) \in [0, \infty)$, finding a lower bound of the former is the same as finding a lower bound of (10). It can be shown that

$$\eta_k(\boldsymbol{\xi}) \geq \frac{\left|\|\boldsymbol{P}_{\boldsymbol{U}_S}^{\perp}\boldsymbol{a}_k\|_2 - \|\boldsymbol{P}_{\boldsymbol{U}_S}^{\perp}\boldsymbol{\xi}\|_2\right|}{\|\boldsymbol{P}_{\boldsymbol{U}_S}\boldsymbol{a}_k\|_2 + \|\boldsymbol{P}_{\boldsymbol{U}_S}\boldsymbol{\xi}\|_2} \tag{11a}$$

$$= \frac{\left|\|\boldsymbol{P}_{\boldsymbol{U}_S}^{\perp}\boldsymbol{a}_k\|_2 - \|\boldsymbol{P}_{\boldsymbol{U}_S}^{\perp}\boldsymbol{\xi}\|_2^2\right|}{\|\boldsymbol{P}_{\boldsymbol{U}_S}\boldsymbol{a}_k\|_2 + \sqrt{\|\boldsymbol{\xi}\|_2^2 - \|\boldsymbol{P}_{\boldsymbol{U}_S}^{\perp}\boldsymbol{\xi}\|_2^2}}, \tag{11b}$$

$$\geq \frac{\left|\|\boldsymbol{P}_{\boldsymbol{U}_S}^{\perp}\boldsymbol{a}_k\|_2 - \|\boldsymbol{P}_{\boldsymbol{U}_S}^{\perp}\boldsymbol{\xi}\|_2^2\right|}{\|\boldsymbol{P}_{\boldsymbol{U}_S}\boldsymbol{a}_k\|_2 + \sqrt{\delta^2 - \|\boldsymbol{P}_{\boldsymbol{U}_S}^{\perp}\boldsymbol{\xi}\|_2^2}}, \tag{11c}$$

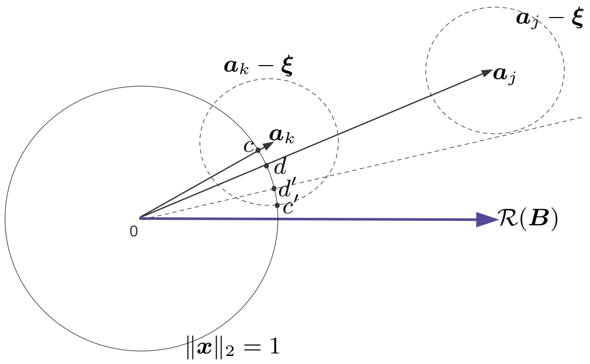

Fig. 1: An illustration of the RMUSIC idea. In this figure, we assume that $k \in \Lambda$ and $j \notin \Lambda$; $\boldsymbol{a}_k$ is perturbed by modeling error so that it lives outside $\mathcal{R}(\boldsymbol{B})$. The distances between points c, d, c' and d' and the horizontal line representing $\mathcal{R}(\boldsymbol{B})$ are $\gamma_{\rm MUSIC}(k)$, $\gamma_{\rm MUSIC}(j)$, $\gamma_{\rm RMUSIC}(k)$ and $\gamma_{\rm RMUSIC}(j)$, respectively.

where (11a) is by the triangle inequality, and (11c) results from the constraint $\|\boldsymbol{\xi}\|_2 \leq \delta$. Let $\theta = \|\boldsymbol{P}^{\perp}_{\boldsymbol{U}_S}\boldsymbol{\xi}\|_2$, which satisfies $\theta \in [0, \delta]$. The minimization of (11c) over $\|\boldsymbol{\xi}\|_2 \leq \delta$ can be simplified to

$$\eta_k^\star = \min_{0 \leq \theta \leq \delta} \frac{\left| \|\boldsymbol{P}^{\perp}_{\boldsymbol{U}_S}\boldsymbol{a}_k\|_2 - \theta \right|}{\|\boldsymbol{P}_{\boldsymbol{U}_S}\boldsymbol{a}_k\|_2 + \sqrt{\delta^2 - \theta^2}}. \quad (12)$$

Problem (12) can be solved easily by a line search algorithm, e.g., bisection. After obtaining $\eta_k^\star$, we substitute $\eta_k^\star$ into (9) to get an approximation of $\gamma_{\rm RMUSIC}(k)$:

$$\hat{\gamma}_{\rm RMUSIC}(k) = \frac{(\eta_k^\star)^2}{(\eta_k^\star)^2 + 1}.$$

Note again that $\hat{\gamma}_{\rm RMUSIC}(k)$ is a lower bound approximation, viz. $\hat{\gamma}_{\rm RMUSIC}(k) \leq \gamma_{\rm RMUSIC}(k)$.

5. SIMULATIONS

We first use a toy example to show the effectiveness of RMUSIC. The signal-to-noise radio (SNR) in this example is 20dB. A subset of the U.S.G.S. library with $K = 257$ spectra is adopted as the correctly calibrated dictionary. $N = 6$ endmembers are randomly picked from this library as the true endmembers; modeling errors with $\epsilon = 0.45$ are manually added to the dictionary members, constructing the available dictionary. The correctly calibrated endmembers and the corresponding available library emembers are shown in Fig. 2. We set $\delta = 1.1\epsilon$ in this example. The signal subspace $\mathcal{R}(\boldsymbol{U}_S)$ is estimated by HySiMe. Fig. 3(a)-(b) show the identification results of RMUSIC and MUSIC, respectively. We can see that RMUSIC gives good identification: the $\hat{\gamma}_{\rm RMUSIC}(k_n)$'s associated with $k_n \in \Lambda$ are all very small, which can visually be distinguished from the values of most $\hat{\gamma}_{\rm RMUSIC}(j)$'s for $j \notin \Lambda$. As for MUSIC, we observe that $\gamma_{\rm MUSIC}(k_n)$'s for $k_n \in \Lambda$ are in general also small, while some $\gamma_{\rm MUSIC}(k_n)$'s (e.g., the most left one) are not easily distinguished from many other residuals.

Next, we consider a Monte Carlo simulation. We quantify the modeling error level by defining the following *dictionary to modeling error ratio* (DMER):

$$\text{DMER(dB)} = 10\log_{10}\left(\|\boldsymbol{a}_{k^\star}\|_2^2/\epsilon^2\right),$$

where $k^\star = \arg\min_{k=1,\ldots,K} \|\boldsymbol{a}_k\|_2$. The setup of RMUSIC is described as follows. We choose the RMUSIC parameter δ by the following formula

$$\delta = \frac{1-\alpha}{1+\alpha}\|\boldsymbol{a}_{k^\star}\|_2,$$

where $\alpha \in [0, 1]$ is given. The parameter α controls the correlation between the RMUSIC-resulted dictionary member $\boldsymbol{a}_{k^\star} - \boldsymbol{\xi}$ and the original one. Specifically, under $\|\boldsymbol{\xi}\|_2 \leq \delta$, it can be shown that the above choice of δ leads to $\frac{(\boldsymbol{a}_{k^\star} - \boldsymbol{\xi})^T \boldsymbol{a}_{k^\star}}{\|\boldsymbol{a}_{k^\star} - \boldsymbol{\xi}\|_2 \|\boldsymbol{a}_{k^\star}\|_2} \geq \alpha$. Moreover, we have RMUSIC admitting more endmembers than the true number of endmembers, by selecting a number of $N_u > N$ indices in the RMUSIC procedure. The same applies to MUSIC. In practice, such a set of over-admitted endmembers can be used as the pruned dictionary for sparse regression [7]. We consider the the following detection probability

$$\Pr\left\{\Lambda \subset \hat{\Lambda}\right\} \quad (13)$$

where $\hat{\Lambda}$ is an algorithm's extracted dictionary index set, as the performance measurement, since sparse regression works only if $\Lambda \subset \hat{\Lambda}$.

In Fig. 4, we show the index set detection probabilities of MUSIC/RMUSIC under various DMERs. The correctly calibrated dictionary here is identical to that used in the toy example. In each trial, $N = 8$ true endmembers are randomly picked, and the available dictionary is obtained by adding modeling errors which are generated following the zero-mean unit-variance i.i.d. Gaussian distribution; we also scale the modeling errors such that the 2-norms are smaller than or equal to ϵ. We set $\text{SNR} = 35$dB and $N_u = 40$. The results are averaged from 1000 trials. We can see from the figure that MUSIC is sensitive to modeling errors; even under high DMERs, MUSIC is not able to identify all k_n's. Generally, using RMUSIC with $\alpha = 0.85$ and 0.95 both improve the detection probability of Λ under all DMERs. Interestingly, one can see that RMUSIC with $\alpha = 0.75$ admits the highest detection probability in lower DMER range; however, when the DMER is higher, using such a relatively small α leads to a slight performance degradation of RMUSIC. The reason is that smaller α implies that one can change $\boldsymbol{a}_k$'s more significantly with the RMUSIC criterion. Hence, it may confuse several similar $\boldsymbol{a}_k$'s with each other. This observation suggests that a more conservative choice of α should be safer for implementing RMUSIC in practice.

Fig. 5 shows the index set detection probabilities of RMUSIC using different N_u's. The result of MUSIC using $N_u = 60$ is also presented as a benchmark. In this simulation, we fix $\alpha = 0.85$ for RMUSIC; the other settings are exactly the same as those in the previous figure. As expected, by increasing N_u, the detection probabilities of RMUSIC are consistently improved. Note that RMUSIC with $N_u = 40$ already exhibits higher detection probabilities than that of MUSIC with $N_u = 60$ under all DMERs. This, again, verifies the robustness of RMUSIC against model mismatches. From a practical viewpoint, this also suggests that RMUSIC provides a more promising way for dictionary pruning in comparison to MUSIC.

6. CONCLUSION

To conclude, a robust MUSIC algorithm was derived for semiblind dictionary-aided HU in the presence of spectral signature mismatches. Our simulation results showed that the robust MUSIC algorithm provides better endmember identification performance than the conventional MUSIC. As a future work, we will test how robust MUSIC performs in real hyperspectral data.

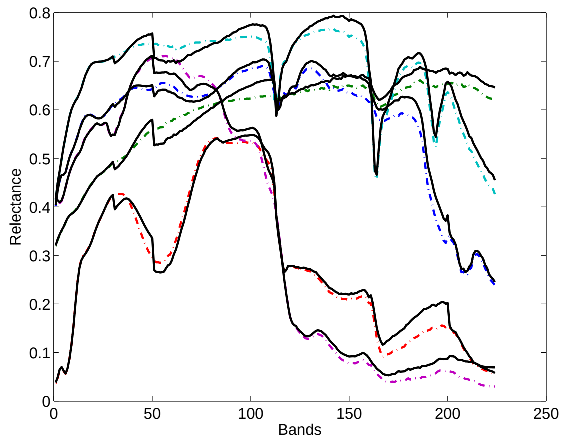

Fig. 2: The calibrated endmembers $\boldsymbol{b}_k$'s (dot lines) and the corresponding available dictionary members $\boldsymbol{a}_k$'s (solid lines).

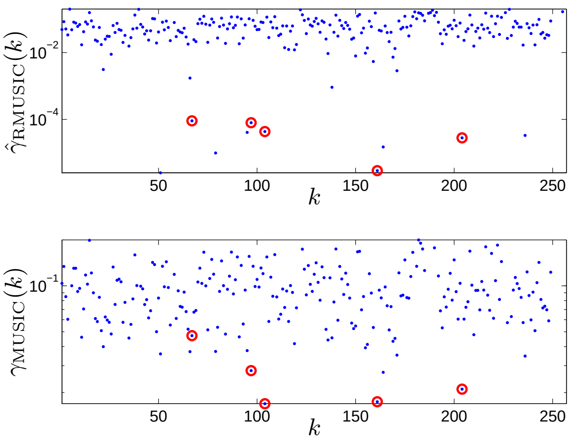

Fig. 3: The approximated RMUSIC projection residuals (upper) and the MUSIC projection residuals (lower) using the example in Fig. 2.

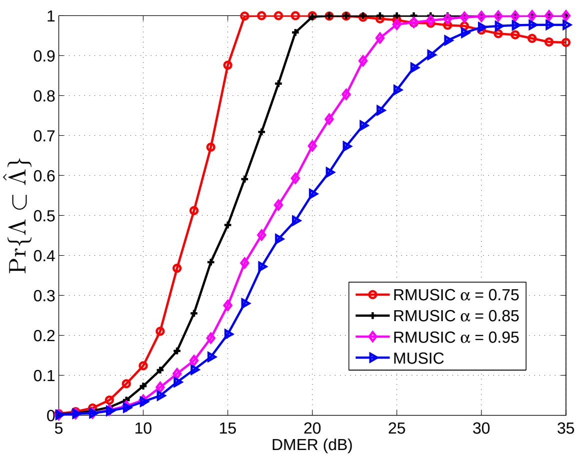

Fig. 4: The $\Pr\{\Lambda \subset \hat{\Lambda}\}$'s of MUSIC and RMUSIC with different α's under various DMERs.

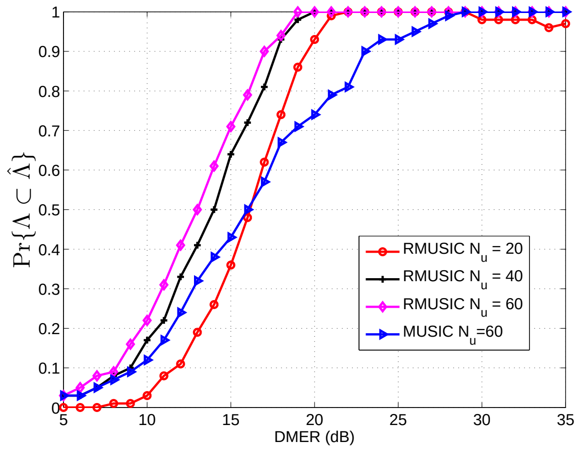

Fig. 5: The $\Pr\{\Lambda \subset \hat{\Lambda}\}$'s of MUSIC and RMUSIC with different N_u's under various DMERs; $\alpha = 0.85$ for RMUSIC.

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